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ON A THEOREM OF MORIMOTO CONCERNING SUFFICIENCY FOR DISCRETE DISTRIBUTIONS[1]

BY L. BROWN

Cornell University

We prove in a discrete setting that if for all test functions, t, there is a $\mathbf{B}$ measurable test function, s, such that $E_p(t) = E_p(s)$ for all $p \in P$ then some subfield of $\mathbf{B}$ is sufficient for P.

The purpose of this note is to call attention to the fact that the conclusion of Theorem 5 of Morimoto (1972) can be strengthened. In the following we use the notation and definitions of Morimoto (1972). In particular, P is a family of discrete distributions on a set X and all subsets of X are measurable. We assume as in Morimoto (1972) that $p(A) = 0$ for all $p \in P$ implies $A = \emptyset$.

The strengthened version of Morimoto's Theorem 5 is as follows:

THEOREM. *Let* $\mathbf{B}$ *be a* σ*-field such that for any test function* $t(x)$ *there is a* $\mathbf{B}$ *measurable test function* $s(x)$ *with* $E(t(x)\,|\,p) = E(s(x)\,|\,p)$ *for all* $p \in P$. *Then* $\mathbf{B}$ *contains a sufficient subfield*, i.e., $\mathbf{B} > \mathbf{B}(\mathbf{M})$.

(Morimoto's conclusion is that $\mathbf{T}(\mathbf{B}) > \mathbf{M}$ which implies that $\mathbf{B}$ is pairwise sufficient, but not that $\mathbf{B}$ is sufficient.)

PROOF. Write $P = \{p_\omega : \omega \in \Omega\}$ where Ω is a well-ordered set. The collection of subsets $(\mathbf{M})$ defined in Morimoto (1972) may be rewritten using a transfinite induction as

$$\mathbf{M} = \{T_{\omega i} : i = 1, 2, \cdots, I_\omega \leqq \infty\} \tag{1}$$

where

$$p_\omega(x) > 0 \quad \text{for all} \quad x \in \bigcup_{i=1}^{I_\omega} T_{\omega i} \quad \text{and} \quad P_{\omega'}\Big(\bigcup_{\omega \leqq \omega'} \bigcup_{i=1}^{I_\omega} T_{\omega i}\Big) = 1\,, \tag{2}$$

and the sets $T_{\omega i}$ are mutually disjoint. (The statement $I_\omega \leqq \infty$ above is intended to mean that the index set $\{i\}$ is countable, but possibly infinite.)

Now, $V \in \mathbf{B}(\mathbf{M})$ if and only if V may be written

$$V = \bigcup_{\omega \in \Omega_0} \bigcup_{i \in I(\omega)} T_{\omega i} = \bigcup_{i=1}^{\infty} \bigcup_{\omega \in \Omega_i} T_{\omega i}$$

where $\Omega_j \subset \Omega$, $j = 0, 1, \cdots$. In order to prove that $\mathbf{B} > \mathbf{B}(\mathbf{M})$ it therefore suffices to prove that any set of the form

$$Q = \bigcup_{\omega \in \Omega'} T_\omega$$

satisfies $Q \in \mathbf{B}$, where $\Omega' \subset \Omega$ and $T_\omega = T_{\omega, i(\omega)}$.

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As a consequence of the characterization described in (7) of Morimoto (1972) and of our definition (2) each set T_ω may be written in the form

$$T_\omega = \{x : p_\omega(x) > 0, p_\xi(x) = 0 \ \ \forall \xi < \omega, \text{ and } p_\omega(x) = k_i p_{\omega_i}(x); i = 1, 2, \cdots, I_{\omega'} \leqq \infty\} \tag{3}$$

where $k_i > 0$ and $\omega_i \geqq \omega$. (Note that the indices ω_i depend on ω, although this is not indicated by the notation.) After defining $0/0 = 0$ we may rewritten (3) as

$$T_\omega = \{x : p_\omega(x)/\textstyle\sum_{i=1}^{I_\omega'} \alpha_i k_i p_{\omega_i}(x) = 1 \ \ \forall \{\alpha_i\} \ni \alpha_i \geqq 0, \sum_{i=1}^{I_\omega'} \alpha_i = 1 \text{ and } p_\xi(x) = 0 \ \ \forall \xi < \omega\}\,. \tag{4}$$

We now prove

LEMMA. *There exists a vector* $(\alpha_1, \cdots)$ *in the simplex defined in* (4) *which satisfies*

$$x \in T_\omega \Leftrightarrow p_\omega(x)/\textstyle\sum_{i=1}^{I_\omega'} \alpha_i k_i p_{\omega_i}(x) = 1 \quad \textit{and} \quad p_\xi(x) = 0 \ \ \forall \xi < \omega\,. \tag{5}$$

PROOF. Consider the metric space, M, consisting of points $(\alpha_1, \cdots, \alpha_i, \cdots)$, $i = 1, \cdots, I_\omega'$, satisfying $0 \leqq \alpha_i \leqq \max(2^{-i}, 2^{-i}k_i^{-1})$; and with metric, ρ, given from the sup (L_∞) norm: $\rho(\alpha, \beta) = \sup\{|\alpha_i - \beta_i| : i = 1, \cdots, I_\omega'\}$. This is a complete metric space.

If $x \notin \bigcup_{i=1}^{I_\omega} T_{\omega i}$ then either $p_\xi(x) = 0$ for all $\xi < \omega$ or $p_\omega(x) = 0$ so that the r.h.s. of (5) cannot hold true.

For given $x \in D_\omega = \bigcup_{i=1}^{I_\omega} T_{\omega i} - T_\omega$ let S_x denote the set of points in M which satisfy $p_\omega(x) \sum_{i=1}^{I_\omega'} \alpha_i = \sum_{i=1}^{I_\omega'} \alpha_i k_i p_{\omega_i}(x)$. For $x \in D_\omega$ $p_\omega(x) > 0$ and there is some index, ω_i^x, say, such that $p_\omega(x) \neq k_i p_{\omega_i^x}(x)$. It follows that if $\beta \in S_x$ and β' satisfies $\beta_i' = \beta_i$ for $i \neq \omega_i^x$ and $\beta'_{\omega_i^x} \neq \beta_{\omega_i^x}$ then $\beta' \notin S_x$. Hence the interior of S_x is empty.

The fact that $\alpha_i \leqq \max(2^{-i}, 2^{-i}k_i^{-1}p_{\omega_i}^{-1}(x))$ and the dominated convergence theorem lead to the conclusion that S_x is closed in M. Since D_ω is countable the Baire category theorem may then be invoked to establish the existence of a point $\alpha' \in M$ such that $p_\omega(x) \sum_{i=1}^{I_\omega'} \alpha_i' \neq \sum_{i=1}^{I_\omega'} \alpha_i' k_i p_{\omega_i}(x)$, for all $x \in D_\omega$. The vector $\alpha'' = \alpha'/\sum_{i=1}^{I_\omega'} \alpha_i'$ is in the simplex described in (4) and satisfies the conclusion of (5). □

Fix any vector in the simplex which satisfies (5). Then $T_\omega = R_\omega - S_\omega$ where

$$R_\omega = \{x : p_\omega(x)/\textstyle\sum_{i=1}^{I_\omega'} \alpha_i k_i P_{\omega_i}(x) \geqq 1 \text{ and } p_\xi(x) = 0 \ \ \forall \xi < \omega\} \subset \bigcup_{i=1}^{I_\omega} T_{\omega i}\,,$$

and

$$S_\omega = \{x : p_\omega(x)/\textstyle\sum_{i=1}^{I_\omega'} \alpha_i k_i p_{\omega_i}(x) > 1 \text{ and } p_\xi(x) = 0 \ \ \forall \xi < \omega\}\,.$$

Note that if $p_\omega(x) > 0$ and $p_\xi(x) > 0$ for some $\xi < \omega$ then $p_\omega(x)/lp_\xi(x) < 1$ for l sufficiently large. Since $\{x : p_\omega(x) > 0\}$ is countable we may thus rewrite R_ω as

$$R_\omega = \{x : p_\omega(x)/[\textstyle\sum_{i=1}^{I_\omega'} \alpha_i k_i p_{\omega_i}(x) + \sum_{i=1}^{J_\omega} l_j p_{\xi_j}(x)] \geqq 1\}$$

where the ξ_j satisfy $\xi_j < \omega$ and the l_j are suitable positive constants and $0 \leqq J_\omega \leqq \infty$. S_ω has a similar expression. Hence

$$Q = \textstyle\bigcup_{\omega \in \Omega'} (R_\omega - S_\omega)\,. \tag{6}$$

Since $S_\omega \subset R_\omega$ for all $\omega \in \Omega'$ and since for $\omega \neq \omega'(\bigcup_{i=1}^{I_\omega} T_{\omega i}) \cap (\bigcup_{i=1}^{I_{\omega'}} T_{\omega' i}) = \emptyset$ we have $R_{\omega'} \cap R_\omega = \emptyset$ for all $\omega, \omega' \in \Omega'$ with $\omega' \neq \omega$. The expression (6) may thus be rewritten as

$$Q = \bigcup_{\omega \in \Omega'} R_\omega - \bigcup_{\omega \in \Omega'} S_\omega . \tag{7}$$

To prove the theorem it therefore suffices to show that any set of the form $\bigcup_{\omega \in \Omega'} R_\omega$ or $\bigcup_{\omega \in \Omega'} S_\omega$ is an element of **B**.

Let $W' = \bigcup_{\omega \in \Omega'} A(p_\omega) = \{x : p_\omega(x) > 0 \text{ for some } \omega \in \Omega'\}$.

Let $t = \chi_{\bigcup_{\omega \in \Omega'} R_\omega}$ and let s be the **B** measurable function generated by the hypothesis of the theorem such that $E(s \mid p) = E(t \mid p)$ for all $p \in P$. $t' = t$ is the essentially unique test function relative to the measure given by $p_\omega + \sum \alpha_i k_i p_{\omega_i} + \sum l_j p_{\xi_j}$ maximizing $E(t' \mid p_\omega)$ subject to the side conditions

$$E(t' \mid p_{\omega_i}) \leqq E(t \mid p_{\omega_i}) \quad \text{and} \quad E(t' \mid p_{\xi_j}) \leqq E(t \mid p_{\xi_j}) ;$$
$$i = 1, \cdots, I_\omega', j = 1, \cdots, J_\omega .$$

Hence $s(x) = t(x)$ for all $x \in W'$.

Since $t(x) = 0$ for $x \notin W'$ we thus have $s(x) \geqq t(x)$ for all $x \in X$. Suppose $s(x) > t(x)$ for some $x \in X$. Then, for some p_ω, $p_\omega(x) > 0$ and $E(s \mid p_\omega) > E(t \mid p_\omega)$, a contradiction.

It follows that $s(x) = t(x)$ for all x so that $\bigcup_{\omega \in \Omega'} R_\omega \in$ **B**. Similarly $\bigcup_{\omega \in \Omega'} S_\omega \in$ **B** (use the fact that **B** is a σ-field). As described above this proves that **B** $>$ **B**(**M**), which is the desired conclusion.

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Statistics Center
Rutgers University
New Brunswick, N. J. 08903